Nonuniform sampling
AN1

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1 Introduction

To process signals digitally, they obviously have to be presented in the appropriate digital format. Therefore the original analog signal, before processing, has to be converted into a digital one, i.e. it has to be digitized. Once a signal is digitized, the features of the obtained digital signal, good as well as bad, are fixed and nothing can be done to change them. In an ideal world, these features would exclusively depend and actually would copy the features of the original analog signal. The reality is different. The two basic operations of any analog-digital conversion, namely, sampling and quantization, impact the characteristics of the digital signal substantially.

The characteristics of the analog signal at the ADC input and of the digital signal at its output are just similar rather than identical. How large and significant the differences between them are depends on the digitizing methods and their implementations applied.

Let us illustrate that by two simple examples:

- Aliasing or totally indistinguishable representation of different frequencies by one and the same data set might occur only in result of periodic sampling. When sampling is nonuniform, frequencies are represented by a digital signal in a unique way.

- Spurious frequencies appearing in spectra of digital signals are produced by the traditional fixed threshold quantization. To remove them, either sampling or quantization has to be irregularized or randomized.

This application guide describes the sampling procedure based on the use of nonuniform (non-equidistant) sampling. The application of randomized quantization is described in a separate application guide [1].

2 Sampling

Information carried by an analog signal can be represented in a digital form by a sequence of its instantaneous values measured at discrete time instants. These signal readings are usually considered as signal sample values and the process of taking them is referred to as sampling. The instants at which the samples are obtained form a stream of events, which can be depicted graphically as a sampling point process. Characteristic features of the sampled signals to a large extent depend on the patterns of the point processes generated and used for sampling. When sampling is mentioned in the context of DSP, usually it is assumed that it is deterministic and uniform (equidistant). The model of sampling according to which signal samples are separated by time intervals with a constant and known duration is the most popular. This is readily comprehensible because such a sampling approach appears to be the most natural and obvious. It also has a number of attractive advantages.
However, as was established relatively long ago, the application of periodic sampling alone does not suffice. The periodic sampling model is not applicable when fluctuations in sampling instants cannot be ignored or when signal samples can be obtained only at nonuniform or even random time intervals. Studies have indicated that randomness in sampling is not always harmful; random irregularities in sampling sometimes can even be beneficial. These irregularities, if properly introduced, provide, for instance, such a useful effect as the suppression of aliasing. And such sampling itself usually is considered as nonuniform.

Depending on the required performance specification of the signal processing system to be developed, given both in functional and performance quality terms, sampling can be adapted by taking the following decision:

- either periodic or nonuniform sampling has to be chosen. While periodic sampling is preferable, in high frequency signal processing cases, the required periodic sampling rate might be too high. Then usage of nonuniform sampling might be better. However special and more complicated signal processing algorithms have to be used in that case. Therefore adapting the sampling operation to the specific signal processing conditions comes down to a trade-off between the complications caused either by high sampling rate or more complex algorithms.

**Periodic sampling** is preferable whenever the spectrum of the signal can be restricted as required by the Sampling Theorem. First, periodic sampling is the simplest method of performing this procedure and is easy to implement. Second, periodic sequences of signal samples are well suited to digital processing. Note only that, many highly efficient fast algorithms are applicable.

**Randomized sampling** may prove to be more profitable when it is undesirable or even impossible to pre-filter signals before their analog digital conversion, when the signal to be processed contains components at frequencies exceeding half of the sampling rate acceptable under the given specific conditions. However the essence of randomized sampling is taking the signal sample values at unknown random time instants. Therefore application of randomized sampling is limited to the relatively seldom-met cases where the information about the exact sampling instants is not relevant.

**Pseudorandomized sampling** is the most often used nonuniform sampling based anti-aliasing technique. The indications for its usage are the same as for the randomized sampling except that the sampling instants in this case are known with high precision.

The significance of the sampling operation is determined by the fact that many essential digital signal characteristics, impacting the whole signal processing process substantially, depend on the methods and techniques used to perform it.

In the traditional DSP case, the only way to reduce this often undesirable dependence is increasing the sampling frequency. However, the possibilities then are poor and limited. In addition, this approach in many cases produces an increased number of bits requiring more complicated hardware for the subsequent processing of the obtained in this way digital signals.

Deliberate introduction, when necessary, of an element of randomness into the sampling operation helps a lot in obtaining the flexibility essential for adaptation of the analog-digital conversion to the conditions of the given specified signal processing
Therefore the application of nonuniform sampling is more flexible than use of traditional uniform sampling.

3 Principles of nonuniform sampling

The properties of nonuniformly sampled signals are mainly defined by the mode of generation of point streams used for the implementation of sampling, that is, the selection of points in time for signal readout.

Although there are a relatively large variety of known non-equidistant spaced point processes, only a few of them actually have the characteristics required for high-performance sampling. The available expertise in application of various random point streams suggests that the most advisable technique for producing sampling instants is based on additive random point process. It is really well suited for the purposes of deliberate sampling randomization. This point process has such remarkable properties and is so flexible that it suits random sampling applications very well. Specifically, sampling carried out in this way ensures that all parts of any input signal are sampled with equal and constant probability. Therefore such sampling is signal-independent. And it provides for unbiased (no systematic errors) estimation of signal parameters, including spectral parameters.

However, there are several other point processes which should also be considered because they are connected to relatively frequently observed and important effects such as fluctuation of uniform sampling instants or random selection of uniformly spaced samples.

3.1 Uniform sampling with jitter

Fluctuation of sampling instants is a fairly common occurrence. It can even be said that it is always present. In the case of uniform sampling with jitter, signal samples \( \{ t_k \} \) are taken at time moments \( t_k = kT + \tau_k, T > 0 \), where \( T \) is a period of uniform sampling, but \( \{ \tau_k \} \) is a family of identically distributed independent random variables with zero mean.

This sampling scheme is illustrated in Figure 1. It shows the probability density functions \( p_k(t) \) of time intervals \( (t_k - t_0) \). As can be seen from Figure 1d, this particular function has multiple peaks. Note that as \( t \) increases the peaks shown do not decrease.

To understand the meaning of the function \( p(t) \), imagine that a narrow time window \( \Delta t \) is moved along the \( t \)-axis. Under the condition \( \Delta t \to 0 \), the function \( p(t) \) at any arbitrary time instant is equal to the probability that one of the sampling points will fall within this window. Therefore if a signal is sampled at the instants which are determined by the statistical relationship illustrated by Figure 1d some parts of the signal will be sampled with a higher probability than others. This is obviously undesirable, as it will lead to the signal processing errors.

There is an exception. If the time intervals \( (t_k - t_0) \) are distributed uniformly in the intervals \( (kT \pm 0.5T) \) respectively, then the resulting sampling point density function is constant. But in fact this method of generating nonuniform sampling point stream has a number of substantial disadvantages, which prevent its wide application:
• The random variables \( \{ \tau_k \} \) should be distributed strictly uniformly within the given intervals
• Time intervals between any two sampling instants may be very short.

![Figure 1.](image)

**Figure 1.** Probability density functions characterizing periodic sampling with jitter. a), b), c) probability density functions of a sum of 1, 2 and 3 time intervals, respectively; d) resulting sampling point density function.

### 3.2. Additive nonuniform sampling

In the case of additive nonuniform sampling, signal samples are taken at time moments \( t_{k+1} = t_k + \tau_k \), where \( \{ \tau_k \} \) is a family of identically distributed independent positive random quantities. Such a nonuniform point stream can be easily implemented to suppress the overlapping effect. The degree of randomization can be varied through appropriate selection of only one parameter - \( \sigma / \mu \), where \( \mu \) and \( \sigma \) are the mean value and standard deviation of the intervals between sampling points, respectively. Obviously the mean sampling rate is equal to \( 1 / \mu \).

The additive random sampling scheme itself is illustrated by Figure 2. Probability density function of time interval \( (t_k - t_0) = \tau_1 + \tau_2 + \ldots + \tau_k \) in this case can be calculated as \( p_k(t) = p_{k-1}(t) \ast p_1(t) \), where the asterisk \( \ast \) denotes the composition operation, and \( p_1(t) = p_1(t) \) is a density function of \( \{ \tau_1 \} \) distribution. On the grounds of central limit theorem in statistics:

• As the random variable \( (t_k - t_0) = \tau_1 + \tau_2 + \ldots + \tau_k \) represents the net result of a linear sum of statistically independent constituent variables, then whatever probability density function these variables may have, the probability density

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\( (t_k - t_0) = \tau_1 + \tau_2 + \ldots + \tau_k \) will approach the normal form as \( k \) approaches infinity.

As can be seen from Figure 2, the sampling point density function \( p(t) = \sum_{k=1}^{\infty} p_k(t) \) in the case of additive nonuniform sampling with \( t \) increasing will always tend to become flat. The value of this constant level is equal to \( 1/\mu \).

When nonuniform sampling is selected for application in a given case, an appropriate sampling rate has to be set up. The criteria for the choice of an appropriate sampling rate, in the case of nonuniform sampling, completely differ from those commonly used for periodic sampling. The spectral component of the signal with highest frequency to be nonuniformly sampled and processed actually is not a criterion then. The mean sampling rate is calculated by evaluating the number of signal samples needed and the longest time interval during which those samples have to be acquired for one signal realization. While the minimum of the signal samples to be taken at periodic sampling actually is equal to the required number of them in the case of nonuniform sampling, excessive samples often are taken at periodic sampling just to avoid aliasing.

![Figure 2](image)

**Figure 2.** Probability density functions characterizing additive random sampling. a), b), c) probability density functions of a sum of 1, 2 and 3 time intervals, respectively; d) resulting sampling point density function.
4 Advantages

4.1. Avoiding aliasing

Assume that a digital data set, representing a signal sample value sequence, is given. It is shown graphically in Figure 3. These signal samples might be used for reconstruction of the original sinusoidal signal they belong to. The sine function 1 (black) is found to be fitting the data. However, if the reconstruction process is continued, it becomes clear that there are other sinusoids at differing frequencies, which can be drawn exactly through the same sample value points like the first one (dotted curves). All these sinusoids are aliases and overlapping of them is aliasing.

Aliasing results in an uncertainty. Indeed, look at these sine waves of different frequencies. Which of them is the right one? Evidently impossible to say if no additional information is supplied.

The effect of aliasing, of course, is well known. As well as the means how to avoid it. It has been proved that there will be no uncertainty if all spectral components above a certain frequency are filtered off the original signal by a lowpass filter. (Under certain conditions aliasing would not occur if the bandwidth of the signal does not exceed half of the sampling frequency no matter how high is the central frequency of the signal). Then it is needed to sample with frequency at least twice higher than the highest spectral component present in the signal. However, while the satisfaction of this requirement resolves the aliasing problem, such compulsory filtering off of the signal components with upper frequencies also imposes a very drastic limitation on application of digital signal processing in frequency domain. The achievable bandwidth of digital systems then is determined by the achievable sampling rate. The latter obviously depends on the microelectronic techniques used. To widen the bandwidth, more costly and more power consuming chips have to be used.

![Figure 3](image-url)  
**Figure 3.** Illustration of the aliasing effect taking place in the case of periodic sampling.

It is very tempting to do something and to eliminate this limitation. That would open up a broad area of new exciting digital signal processing applications. Apparently, if there would be some other way how to avoid aliasing, digital processing of signals would be applicable in a much broader frequency range. So the question is: is there
such a possibility? Fortunately, the answer is affirmative. Application of nonuniform sampling offers this.

To see how nonuniform sampling helps in avoiding aliasing, look at Figure 4. The lower frequency sine function is again sampled and the corresponding data set is obtained. However the distances between the sampling instants along the time axis differ. And that proves to be very useful. Indeed, it can be easily seen that only one sine function can be drawn exactly through the indicated points representing the sample value sequence taken from the first sinusoid. Other frequencies simply do not go through them.

![Figure 4](image_url)

**Figure 4.** Only one sine function goes exactly through the nonuniformly spaced sample values.

This effect can be easily checked. Studies of it would confirm the fact that, in the case of correctly performed nonuniform sampling, sinusoids with different frequencies are represented by different data sets. Therefore nonuniformly sampled signals have no completely overlapping aliases like those observed at periodic sampling. Consequently, it can be expected that application of nonuniform sampling should open up the possibility of distinguishing all spectral components of the signal, even if their frequencies substantially exceed the mean sampling rate.

### 4.2. Application example: oscillogram

The nonuniform sampling is well applicable for obtaining oscillograms in cases where repetitious signals are observed. It allows to view and measure waveform characteristics of signals in a spectral region which more times exceeds the value of mean sampling rate. Let us illustrate that by an example. The very simple procedure based on overlapping of signal samples, taken from the number of consecutive signal periods, to the one signal period. Figure 5 illustrates the oscillograms obtained by such a procedure in two cases – when uniform and nonuniform sampling of signal is used. A waveform reconstructed from uniformly sampled signal is shown in Figure 5a, while a waveform, reconstructed from sample values of a pseudorandomly sampled signal, is shown in Figure 5b. An additional linear interpolation is applied for better visual perception.

As can be seen from the oscillogram form from uniformly sampled signal, the signal samples are concentrate into few points. The number and position of these points depends on the ratio between signal and sampling period. In the case of additive nonuniform sampling the sampling point density function is constant. That means that
the point density function of oscillogram also is constant and does not depends on the signal and sampling frequencies.

The waveform obtained in the oscillogram is fuzzy. There are two reasons for that. First, the discrete signal usually is corrupted by some noise, for instance, by quantization noise. The value of SNR is 20 dB in the presented example of oscillogram. Second, the fuzziness of the oscillogram is due to fluctuations or jitter of the sampling instants. The standard deviation of jitter is \(~1\%\) of mean sampling interval between two samples. Special smoothing procedure can be used to remove this fuzziness in oscillogram.

![Figure 5](image)

**Figure 5.** Example of oscillograms. The frequency of the highest signal component 10.3 times exceeds the mean sampling rate a) uniform sampling; b) additive nonuniform sampling.

### 4.3. Application example: spectrogram

The term spectral analysis covers a wide area of diverse signal analysis techniques. When signal spectra are considered on the basis of complex exponential functions, the discrete Fourier transform (DFT) is the most versatile analytical tool. The expectation of the spectrum of a properly randomly sampled signal coincides with the spectrum of the respective analog signal even if the mean sampling rate is considerably below the upper frequency of the signal spectrum. Thus nonuniformity of sampling for spectral analysis applications seems to be very useful, as the following example indicates.
Suppose that a spectrogram of a composite wideband signal (few sinusoids and noise) has to be estimated. The Figure 6 shows the spectrogram results obtained by very simple formula \[ S(f) = \frac{1}{\Theta} \left| \sum_{n=0}^{N} x_n \exp(-j2\pi f t_n) \right|^2 \]. The spectrograms given there display the spectrum of a noisy signal containing three sinusoidal components at the indicated frequencies. The Signal to Noise Ratio (SNR) is 10 dB. In periodic sampling \((t_n = nT)\) case (Figure 6a) there are aliases for signal components. The second spectrogram (Figure 6b) was obtained simply by changing the sampling mode from uniform to additive nonuniform. As can be seen, nonuniformity of sampling in this case efficiently improves the spectrogram, allowing to determine true signal components at frequencies exceeding the half of mean sampling rate.

The increased background noise level in the second spectrogram is the disadvantage of the use of nonuniform sampling. To suppress it the more complicated signal processing procedures should be applied for solving of spectral analysis tasks.

![Figure 6. Example of spectrograms of composite noisy signal. a) uniform sampling; b) additive nonuniform sampling.](image-url)
5 Implementation specifics

5.1 Pseudorandomized sampling

In general, each of the taken signal samples has to be represented by two numbers: the value of the sample and the time instant when it has been taken. There are no problems for periodic sampling as the sampling intervals are constant. Only the sample values are measured then and timing is taken care of just by counting the signal samples taken.

The situation is quite different with nonuniform sampling. The duration of sampling intervals in that case is varying and each sampling event takes place at a specific time instant. How to obtain the information exactly when each signal sample has been taken is quite a demanding engineering problem. Especially if the required time resolution is taken into account. Indeed, the period, for example, of 1 GHz signal is 1 nanosecond. To sample such a signal, the time resolution obviously has to be a few picoseconds. More information about practical implementation of such sampling can be found in [2].

The nonuniform sampling timing technique is based on taking the signal samples at exactly predetermined time instants stored in memory.

A typical sample value block is shown in Figure 7. As can be seen, the distances between the signal samples vary. However, the sampling process is well controlled. Each sampling event takes place at a time instant determined prior to sampling.

![Figure 7](image.png)

Figure 7. A block of nonuniformly taken samples.

The sampling time intervals are given, apparently, by finite digital numbers. Therefore the value of the least significant bit, measured in time units, represents the discreteness of the pseudorandomized sampling process or, in other words, the period of the hidden periodicity present in this process.

Due to this periodicity, aliasing is observed also in the case of pseudorandomized sampling. However the frequency determining aliasing then is the frequency of this hidden periodicity rather than the mean sampling frequency. As the former typically exceeds the latter many times, the usage of pseudorandomized sampling usually leads to avoidance of aliasing in the classical sense of this term.
The possible approach of generating pseudorandom pulse streams is illustrated in Figure 8. The pseudorandom pulse stream is formed by taking each $n_i$-th pulse from the high-frequency periodic pulse sequence generated by stabilized clock. The variables $n_i$, which are actually pseudorandom, are generated by two pseudorandom number generators PRNG 1 and 2. These pseudorandom numbers are fed into one of the two counters. This counter is switched to the high-frequency stabilized clock generator that begins to count the clock pulses. Depending on how this counter has been preset it is sooner or later filled to its capacity. When the counter is overfilled, a pulse for the output sequence is formed. The next pulse is obtained by repeating this cycle with the other counter and PRNG.

The most valuable property of such a pseudorandom pulse stream is, that it is fully determined by the clock frequency and by the sequence of pseudorandom numbers used. This means that:

- Such pulse streams can be reproduced at will whenever this is necessary.
- The each pulse of stream will always appear at an exactly predetermined instant.
- Descriptions of particular pseudorandom pulse stream can be stored in computer memory and this information then used for calculating the time instants when it is necessary.

![Block diagram of the pseudorandom pulse streams generator.](image)

### Figure 8. Block diagram of the pseudorandom pulse streams generator.

The listing of MATLAB function is given in the Appendix 1. This function is provided for simulation of nonuniform sampling. The possible use and parameters description is done in the help section of function.
5.2 Impact of sampling jitter

As it was shown, the randomness present at sampling might play a significant positive role. However that is not the case always. Random fluctuations of sampling instants or jitter might impact the analog-digital conversions and the subsequent signal processing quite strongly in a bad way. And such jitter is a fairly common occurrence. It can even be said that jittering happens always with more or less noticeable harmful effects due to it. How seriously jitter affects the precision of processing results depends on the kind of processing being performed and, of course, on the magnitude of the sampling instant fluctuations.

The essence of the harmful sampling instant jitter is random deviations of the real sampling instants from their corresponding preset values. When a sample value is taken from a signal at a time instant a little bit earlier or later than the predetermined instant, which is used at calculations, an error is introduced at calculations. To see how sensitive algorithms can be to such sampling instant fluctuations, look at Figure 9.

![Figure 9. Spectral analysis of a signal sampled using the additive pseudorandom sampling, number of processed samples $N=4096$: without jitter; (b) jitter with standard deviation 20 psec, mean sampling rate 80 Msamples/s.](image-url)
As it can be seen from Figure 9, jitter error introduces rather strong background noise into the spectral analysis results. Experiments have shown that, in practice, sampling jitter presents the major error source of signal analysis. Therefore electronic devices performing signal digitizing on the basis of nonuniform sampling have to ensure sufficiently precise timing. At present moment, that is possible for the frequency band up to few gigahertz.

6 References

Appendix 1. Listing of the MATLAB function for calculation of time instants in the case of nonuniform sampling.

```matlab
function t = tnonunif(N, F_s, SM, mode, t0, option, J);
% TNONUNIF  Generation of time instants for nonuniform sampling.
% t=tnonunif(N, F_s, SM) produces N nonequidistantly spaced time instants with mean sampling rate F_s and ratio SM between mean value and standard deviation of the time intervals.
% If SM=0, the uniform sampling point stream is generated.
% t=tnonunif(N, F_s, SM, 'mode') Nonuniformity can be based on two different calculation ways:
% mode='per' the approach of periodic point stream with jitter is used;
% mode='add' the approach of additive random point stream is used.
% Default mode is 'add'.
% t=tnonunif(N, F_s, SM, 'mode', t0)
% Parameter t0 determine the value of first time instant t(0), the default value is t(0)=0;
% t=tnonunif(N, F_s, SM, 'mode', t0, option)
% Two options for random number distribution are possible:
% option='unif' - time intervals are uniformly distributed;
% option='norm' - time intervals are normally distributed;
% Default option is 'unif'.
% t=tnonunif(N, F_s, SM, 'mode', t0, option, J)
% J - the random number generator is set to its J-th state.
% It allows to reduplicate the generated sampling point stream.
error(nargchk(3,7,nargin));
if nargin<4,  
  mode = 'add';
elseif ~strcmp(mode, 'per')&~strcmp(mode, 'add');
  error('invalid mode');
end
if nargin>5
  if ~strcmp(option, 'unif')&~strcmp(option, 'norm');
    error('invalid distribution option');
  end
else
  option= 'unif';
end
if nargin>6
  v=version;
  if v(1) > 4
    rand('state', J);
    randn('state', J);
  else
    rand('seed', J);
    randn('seed', J);
  end
end
```
t(1)=0;

if strcmp(option, 'unif');
    if strcmp(mode, 'per');
        t=[1:N]./F_s+sqrt(6)*SM./F_s*(rand(1,N) - .5);
        t=t-t(1);
    elseif strcmp(mode, 'add');
        for ii=2:N,
            t(ii)=t(ii-1)+1./F_s+sqrt(12)*SM./F_s*(rand - .5);
        end
    end
elseif strcmp(option, 'norm');
    if strcmp(mode, 'per');
        t=[1:N]./F_s+sqrt(2)*SM./2./F_s*(randn(1,N));
        t=t-t(1);
    elseif strcmp(mode, 'add');
        for ii=2:N,
            t(ii)=t(ii-1)+1./F_s+SM./F_s*(randn);
        end
    end
end

if nargin>4
    t=t+t0;
end