SIGNAL SAMPLING ACCORDING TO TIME-VARYING BANDWIDTH

R. Shavelis and M. Greitans

Institute of Electronics and Computer Science
14 Dzerbenes Str., Riga LV-1006, Latvia

Introduction

The Nyquist-Shannon sampling theorem states that every bandlimited signal is uniquely determined by its samples taken uniformly at a rate of at least twice the bandwidth of the signal. The bandwidth follows from analyzing the signal by Fourier transform (FT) in the whole duration of the signal and thus can be considered as the global bandwidth. Such analyze, however, does not provide information about time-varying frequency content of the signal, which in turn could be used to sample the signal in signal-dependent way with more samples taken at high frequency regions and less samples – in low frequency regions.

Considering this signal-dependent sampling problem, we propose an extended sampling theorem, which states that signals can be sampled non-uniformly and then perfectly reconstructed if their spectrums obtained by an extended Fourier transform (EFT) are bandlimited. Since, according to the theorem, the sampling instants are determined by the function used in EFT, the aim is to find such a function which reflects the time-varying (local) bandwidth of the signal. This, in comparison to uniform sampling, allows reducing the number of samples required to represent non-stationary signals.

1 Extended sampling theorem

In [1] definitions of extended Fourier transform (EFT) and inverse EFT are given:

$$S(\omega_g) = \tilde{F}[s(t), g(t)] = \int_{-\infty}^{\infty} \frac{s(t)}{g(t)} e^{-j\omega_g m(t)} dt \tag{1}$$

and

$$s(t) = \tilde{F}^{-1}[S(\omega_g), g(t)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega_g) e^{j\omega_g m(t)} d\omega_g, \tag{2}$$

where $S(\omega_g)$ is the EFT spectrum of $s(t)$, $g(t)$ is a positive function and $m(t) = \int_0^t \frac{1}{g(\tau)} d\tau$.

Proposition of the extended sampling theorem: every bandlimited to $[-\Omega_g, \Omega_g]$ signal $s(t)$ is uniquely determined by its samples $s(t_n)$ taken at instants $t_n = m^{-1}(n\Upsilon)$ with a sampling step $\Upsilon \leq \frac{\pi}{\Omega_g}$. The reconstruction formula is

$$s(t) = \sum_{n=-\infty}^{\infty} s(t_n) \mathrm{sinc}(\frac{\pi}{\Upsilon}(m(t) - m(t_n))) \tag{3}$$

In this theorem, the signal is bandlimited if the spectrum $S(\omega_g)$ is zero outside the band $[-\Omega_g, \Omega_g]$. The special case when $m(t) = t$ leads to classical uniform sampling.

Example: EFT of $s_1(t) = \sum_{l=1}^{L} A_l \cos(k_l m(t))$ with coefficients $0 < k_1 < k_2 < \cdots < k_L$ is $S_1(\omega_g) = \sum_{l=1}^{L} A_l \pi(\delta(\omega_g + k_l) + \delta(\omega_g - k_l))$. It follows the signal is bandlimited and can be represented by samples taken at $t_n = m^{-1}(n\Upsilon)$, $\Upsilon = \pi/k_L$ (Figure 1).

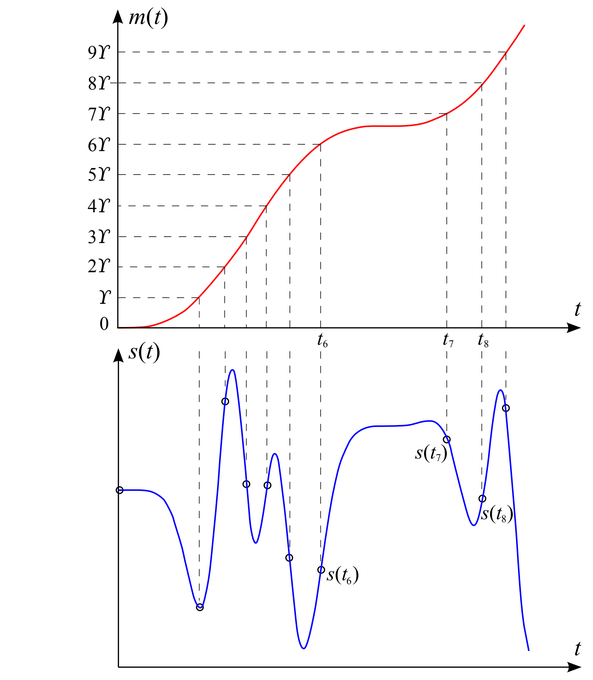


FIGURE 1: Sampling according to $m(t)$: samples $s(t_n)$ ar taken at $t_n = m^{-1}(n\Upsilon)$.

2 Signal-dependent sampling

Aim is to find such function $\hat{f}_{max}(t)$ which reflects the time-varying bandwidth of the signal. Given this function, $m(t)$ is found as

$$m(t) = 2\pi \int_0^t \hat{f}_{max}(\tau) d\tau \tag{4}$$

and signal samples are taken at $t_n = m^{-1}(n\Upsilon)$ with sampling step Υ determined by the bandwidth of EFT spectrum of the signal.

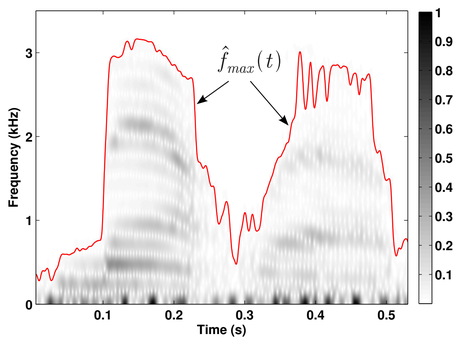


FIGURE 2: Example of $\hat{f}_{max}(t)$ obtained from TFR of the speech signal.

3 Maximum instantaneous frequency

Given a signal

$$s(t) = \sum_{l=1}^{L} A_l \cos(\Phi_l(t)) \tag{5}$$

with constant amplitudes A_l and monotonically increasing functions $\Phi_l(t)$, a maximum instantaneous frequency $f_{max}(t)$ of $s(t)$ is defined as having values

$$f_{max}(\tau) = \max(f_1(\tau), f_2(\tau), \ldots, f_L(\tau)) \tag{6}$$

at any given $t = \tau$. Functions $f_l(t)$ are instantaneous frequencies of cosines in (5).

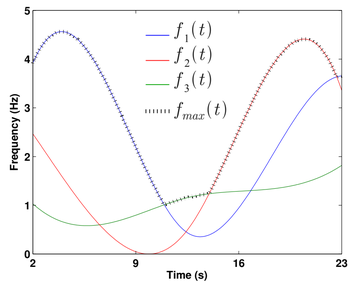


FIGURE 3: Maximum instantaneous frequency $f_{max}(t)$.

4 Estimation of maximum instantaneous frequency

If no frequencies $f_l(t)$ are given or the signal differs from (5), then the idea is to decompose the signal into a finite number of Intrinsic Mode Functions (IMFs) using the Empirical Mode Decomposition (EMD) [2]. In result the first IMF is found and a smoothed version of its instantaneous frequency (IF) is then assumed to be the estimate of the instantaneous maximum frequency of the signal $s(t)$.

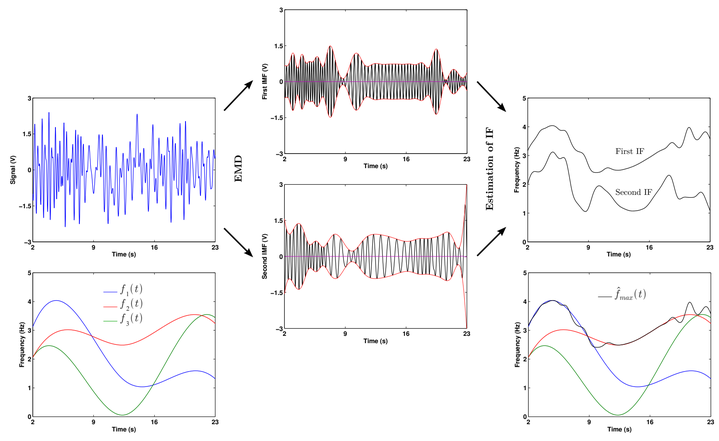


FIGURE 4: Estimation of $\hat{f}_{max}(t)$ of the signal.

5 Numerical results

The first signal (blue line in Figure 5a) is composed of three cosines with time-varying frequencies and the second signal (blue line in Figure 5d) is a real up to 49 Hz low-pass filtered EEG signal.

- The first step before sampling is to estimate the maximum instantaneous frequency of the signal (black lines in Figures 5b and 5e).
- After estimation the sampling follows and the reconstructed signal according to (3) is found. The error signals $s(t) - \hat{s}(t)$ are shown in Figures 5a and 5d by the black lines.
- The precision of reconstruction improves as the sampling step π/Ω_g decreases, i.e., the average number of samples per second increases. This is shown in Figures 5c and 5f by the black lines, while the red lines correspond to uniform sampling case.

FIGURE 5: (a) and (d) – original (blue lines) and error (black lines) signals, (b) and (e) – estimated frequency functions $\hat{f}_{max}(t)$ as black lines, (c) and (f) – RMS values of the error signals depending on the sampling step π/Ω_g, i.e., the average number of samples per second (black lines correspond to signal-dependent sampling case and the red lines – to uniform sampling case).

Conclusions

The results show that adaptation of sampling rate to time-varying spectral content of the signal in the form of estimated maximum instantaneous frequency of the first EMD component allows obtaining better reconstruction from less samples in comparison to uniform sampling. Besides EMD method there may be other techniques like spectrogram analysis or time-varying filtering approach for finding more optimal frequency functions providing more compact EFT spectrums of the signal and thus requiring less samples to represent the signal. This is the topic for further investigation.

References

[1] L. Heyoung and Z.Z. Bien. A variable bandwidth filter for estimation of instantaneous frequency and reconstruction of signals with time-varying spectral content. *IEEE Trans. on Signal Processing*, 59:2052–2071, May 2011.

[2] N.E. Huang and et al. et al. The empirical mode decomposition and the hilbert spectrum for nonlinear and non-stationary time series analysis. In *Proc. R. Soc. Lond. A*, volume 454, pages 903–995, 1998.

This research is supported by ESF project No. 2009/0219/1DP/1.1.1.2.0/09/APIA/VIAA/020 and by Latvian State research program in innovative materials and technologies.

Contacts: modris_greitans@edi.lv
shavelis@edi.lv